n-player threshold participation game

A group of n thieves is deciding between two actions: robbing a casino or entering the labor market and getting a job. If they manage to rob the casino successfully, they split an amount S equally among those who chose to participate in the robbery. If they fail to rob the casino successfully, they obtain a payoff of zero. If they choose to enter the labor market, each of them obtains a payoff of H

In order to rob the casino successfully, at least m thieves are needed, with

$$2 \leq m \leq n$$

1. Assume that
$$\frac{S}{n} > H$$
Find the Nash equilibria

2. Now assume that there exists some $\bar{k}$, with
$$m \leq \bar{k} \leq n$$
such that for every $k \geq \bar{k}$,
$$H > \frac{S}{k}$$
while for every $k < \bar{k}$,
$$\frac{S}{k} > H$$
Find the Nash equilibria

Solution

We characterize the *pure-strategy* Nash equilibria

Let R denote the action “rob the casino” and let W denote the action “work”

If exactly k thieves choose R, then each robber gets

$$u^R(k) = \begin{cases} 0 & \text{if } k < m \\ \dfrac{S}{k} & \text{if } k \geq m \end{cases}$$

Each worker gets

$$u^W = H$$

(1) Case $\frac{S}{n} > H$

We analyze the possible number k of thieves choosing R

If

$$k = 0$$

then everybody works and each thief gets H

A unilateral deviation by one thief to R would lead to

$$k = 1 < m$$

so the deviating thief would get 0, which is worse than H

Therefore, the profile in which everyone works is a Nash equilibrium

Now suppose

$$1 \leq k < m$$

Then every robber gets 0, while by deviating to W she would get H

So no such profile can be a Nash equilibrium

Now suppose

$$m \leq k < n$$

Then each robber gets

$$\frac{S}{k} \geq \frac{S}{n} > H$$

so robbers do not want to switch to W

However, any worker can deviate to R, making the number of robbers equal to $k+1$, and would obtain

$$\frac{S}{k+1} \geq \frac{S}{n} > H$$

So every worker would strictly prefer to join the robbery

Hence, no profile with

$$m \leq k < n$$

can be a Nash equilibrium

Finally, if

$$k = n$$

then everybody robs and each thief gets

$$\frac{S}{n} > H$$

If one thief deviates to W, she would get H, which is strictly lower

Therefore, the profile in which everyone robs is also a Nash equilibrium

Hence, in pure strategies, the Nash equilibria are exactly two: the profile in which all thieves work and the profile in which all thieves rob

(2) Case with threshold $\bar{k}$

We now assume that

$$H > \frac{S}{k} \quad \text{for every } k \geq \bar{k}$$

and

$$\frac{S}{k} > H \quad \text{for every } k < \bar{k}$$

with

$$m \leq \bar{k} \leq n$$

Again, we analyze the possible number k of thieves choosing R

If

$$k = 0$$

then everybody works and gets H

A unilateral deviation to R would generate

$$k = 1 < m$$

so the deviating thief would get 0, which is worse than H

Thus, the profile in which everybody works is a Nash equilibrium

Now suppose

$$1 \leq k < m$$

Then robbers get 0, while by deviating to W they would get H

So no such profile can be a Nash equilibrium

Now suppose

$$m \leq k < \bar{k}$$

Then each robber gets

$$\frac{S}{k} > H$$

so robbers do not want to deviate to W

We must check whether workers want to deviate to R

If

$$k + 1 < \bar{k}$$

then a worker who deviates to R would get

$$\frac{S}{k+1} > H$$

so the deviation is profitable

Therefore, profiles with

$$m \le k \le \bar{k} - 2$$

cannot be Nash equilibria
The only remaining possibility in this range is

$$k = \bar{k} - 1$$

In that case, each robber gets

$$\frac{S}{\bar{k} - 1} > H$$

so robbers do not want to switch to W
If a worker deviates to R, the number of robbers becomes $\bar{k}$, and the deviating worker would get

$$\frac{S}{\bar{k}} < H$$

so workers do not want to join
Hence, whenever

$$\bar{k} - 1 \ge m$$

every profile with exactly $\bar{k} - 1$ robbers is a Nash equilibrium
Finally, suppose

$$k \ge \bar{k}$$

Then each robber gets

$$\frac{S}{k} < H$$

so any robber would prefer to deviate to W
Therefore, no such profile can be a Nash equilibrium
We conclude that:

- if
$$\bar{k} = m$$
then the only pure-strategy Nash equilibrium is the one in which everybody works

- if
$$\bar{k} > m$$
then the pure-strategy Nash equilibria are the profile in which everybody works and all profiles with exactly $\bar{k} - 1$ robbers

Equivalently, when $\bar{k} > m$, there are

$$\binom{n}{\bar{k} - 1}$$

additional equilibria of that second type

Hence, in pure strategies, the Nash equilibria are: always the profile in which all thieves work, and in addition, if $\bar{k} > m$, every profile with exactly $\bar{k} - 1$ thieves choosing R.